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JOHNSVILLE, PENNSYLVANIA

Aeronautical Electronic and Electrical Laboratory

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PHASE REPORT
HEAT FLOW IN RECTANGULAR PLATE OF UNIFORM
THICKNESS UNDER VARYING AMBIENT TEMPERATURE

WEPTASK NO. RAV41J002/2021/R008-03-001
Problem No. 3 (AV-44016)

SUMMARY

An exact solution is found for the temperature distribution in a rectangular plate of uniform thickness under varying ambient temperature. For the special case of temperature variation in the form of a sinusoidal pulse, a numerical comparison is made of the exact solution and an approximate solution obtained by a modification of the solution for constant ambient temperature. The range of validity of this approximate solution is considered in terms of a "rise time."

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Ref: (a) Adams, Edwin P. and Hippisley, R. L. - Smithsonian Mathematical Formulae and Tables of Elliptic Functions, Washington D.C., 1939
(b) Report No. NADC-EL-6143, "Phase Report, Development of Analytical Design of Heat Exchangers for Airborne Electronic Equipment"

INTRODUCTION

The problem of calculating the cooling effect of fins under steady state conditions is relatively elementary, in that it involves the solution of an ordinary differential equation. However, under non-steady state conditions, as when the ambient temperature is varying, a partial differential equation is involved. The solution to this equation must satisfy a specified initial temperature distribution as well as the boundary conditions.

The rate of heat flow into the base of the fin in the steady state may be written as

$$\dot{Q} = (T_b - T_a)F(P,G)$$

where

T_b = temperature at base of fin
T_a = ambient temperature

$F(P,G)$ = a function of the physical and geometrical parameters of the fin.

If now the ambient temperature is a variable, one is tempted to avoid the difficulties associated with the solution of a partial differential equation and to use the above equation, derived under the assumption of steady state conditions, by substituting the variable ambient temperature $T_a(t)$ for the constant ambient temperature T_a, i.e., by writing

$$\dot{Q}(t) = \left[T_b - T_a(t)\right] F(P,G)$$

and

$$Q(t) = F(P,G)\int_0^t \left[T_b - T_a(\tau)\right] d\tau$$

This approximation will be referred to as "quasi-static."

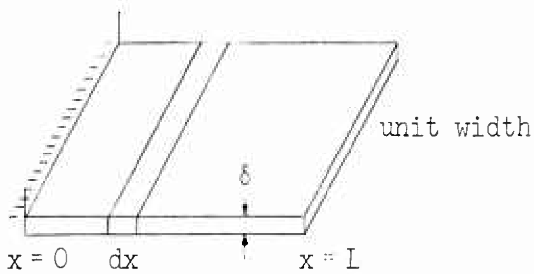
There is no doubt that thi~ procedure furnishes a good approximation if $T_a(t)$ varies slowly enough; but to justify its use the expression "slowly enough" must be made more quantitative.

In this report the general solution of the proper partial differential equation for a fin with fixed base temperature and variable ambient temperature is derived. This is then specialized to the case where the ambient temperature varies as a step function or as a sinusoidal pulse. For this latter case the exact solution is compared with the solution obtained by the approximate method discussed above, and the difference is discussed in terms of a "rise time" obtained from the response to a step function.

For simplicity the fin was assumed to have a uniform thickness (smallest dimension), but the general conclusions are applicable to the case of a tapered thickness.

THEORY

Differential Equation



Assume heat flow in x-direction only. Then,

net rate of heat input to element $= \frac{\partial}{\partial x}(\delta k \frac{\partial T}{\partial x})dx$

rate of heat transfer to ambient medium $= 2h\left[T - T_a(t)\right]dx$

rate of heat accumulation in element $= c\rho\delta \ dx \frac{\partial T}{\partial t}$

$$\therefore \frac{\partial^2 T}{\partial x^2} - \frac{2h}{\delta k}\left[T - T_a(t)\right] - \frac{c\rho}{k}\frac{\partial T}{\partial t} = 0$$

Let $\alpha^2 = \frac{2hL^2}{\delta k}$, $\beta = \frac{c\rho L^2}{k}$, $T_a(t) = T_a + f(t)$; then

$$\frac{\partial^2 T}{\partial (x/L)^2} - \alpha^2 T - \beta\frac{\partial T}{\partial t} = -\alpha^2 \left[T_a + f(t)\right] \tag{1}$$

Boundary and Initial Conditions

At $x = 0$

$$T(t,0) = T_b \tag{2}$$

At $x = L$

$$\frac{\partial}{\partial x}(t,L) = 0 \tag{3}$$

Equation (3) expresses the assumption that the free edge of the fin is insulated, so that no heat flows through it.

When $t = 0$ the temperature distribution assumed is to be

$$T(0,x) = T_a + (T_b - T_a)\frac{\cosh \alpha(1 - x/L)}{\cosh \alpha} \tag{4}$$

This is the steady state temperature distribution with ambient temperature T_a. (This temperature distribution is easily derived from the steady state heat flow equation.)

General Solution by Means of Laplace Transform

Let $\theta(s,x) = \mathcal{L}\{T(t,x)\}$

$F(s) = \mathcal{L}\{f(t)\}$

where

$\mathcal{L}\{\ \} \equiv$ Laplace transform of $\{\ \}$.

Then

$$\mathcal{L}\{\frac{\partial T}{\partial t}\} = s\theta(s,x) - T(0,x)$$

and with the use of (4) the transform of equation (1) becomes

$$\frac{d^2\theta(s,x)}{d(x/L)^2} - (\alpha^2 + \beta s)\theta(s,x) = -\frac{T_a}{s}(\alpha^2 + \beta s) - \alpha^2 F(s) - \beta(T_b - T_a)\frac{\cosh \alpha(1 - x/L)}{\cosh \alpha}$$

The general solution of this ordinary differential equation is

$$\theta(s,x) = A \cosh\sqrt{\alpha^2 + \beta s}\,\frac{x}{L} + B \sinh\sqrt{\alpha^2 + \beta s}\,\frac{x}{L}$$

$$+ \frac{\alpha^2 F(s)}{\alpha^2 + \beta s} + \frac{T_a}{s} + \frac{1}{s}(T_b - T_a)\frac{\cosh \alpha(1 - s/L)}{\cosh \alpha} \tag{5}$$

The transformed boundary conditions become

$$\theta(s,0) = T_b/s \tag{2'}$$

$$\frac{d}{dx}\theta(s,L) = 0 \tag{3'}$$

Substitution of (5) in (2)' gives

$$A = -\frac{\alpha^2 F(s)}{\alpha^2 + \beta s}$$

Substitution of (5) in (3)' gives

$$B = -A \tanh\sqrt{\alpha^2 + \beta s}$$

Thus

$$\theta(s,x) = -\frac{\alpha^2 F(s)}{\alpha^2 + \beta s}\left[\cosh\sqrt{\alpha^2 + \beta s}\,\frac{x}{L} - \tanh\sqrt{\alpha^2 + \beta s}\,\sinh\sqrt{\alpha^2 + \beta s}\,\frac{x}{L}\right]$$

$$+ \frac{\alpha^2 F(s)}{\alpha^2 + \beta s} + \frac{T_a}{s} + \frac{1}{s}(T_b - T_a)\frac{\cosh \alpha(1 - x/L)}{\cosh \alpha}$$

or

$$\theta(s,x) = \frac{\alpha^2 F(s)}{\alpha^2 + \beta s}\left[1 - \frac{\cosh\sqrt{\alpha^2 + \beta s}\,(1 - x/L)}{\cosh\sqrt{\alpha^2 + \beta s}}\right]$$

$$+ \frac{1}{s}\,T_a + (T_b - T_a)\frac{\cosh \alpha(1 - x/L)}{\cosh \alpha}$$

Now

$$T(t,x) = \mathcal{L}^{-1}\{\theta(s,x)\}$$

where

$$\mathcal{L}^{-1}\{\ \} \equiv \text{inverse Laplace transform of } \{\ \}.$$

The first term may be written as $F(s)G(s)$; and if

$$g(t) \equiv \mathcal{L}^{-1}\{G(s)\}$$

there follows from the rule for the inverse transform of a product

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\lambda)g(t - \lambda)d\lambda$$

$$\therefore T(t,x) = \int_0^t f(\lambda)g(t - \lambda)d\lambda + T_a + (T_b - T_a)\,\frac{\cosh \alpha(1 - x/L)}{\cosh \alpha} \tag{6}$$

The inverse transform of $G(s)$ is given by the sum of the residues of

$$G(s)e^{st} = \frac{\alpha^2}{\alpha^2 + \beta s}\left[1 - \frac{\cosh\sqrt{\alpha^2 + \beta s}\,(1 - s/L)}{\cosh\sqrt{\alpha^2 + \beta s}}\right]e^{st}\ .$$

By evaluating $\lim\limits_{s \to -\alpha^2/\beta} (\alpha^2 + \beta s)G(s)$, it is seen that the residue at the pole, $s = -\alpha^2/\beta$, is zero. The expression

$$\frac{-\alpha^2 \cosh\sqrt{\alpha^2 + \beta s}\,(1 - x/L)}{(\alpha^2 + \beta s)\cosh\sqrt{\alpha^2 + \beta s}}e^{st} \equiv \frac{P(s)}{Q(s)}e^{st}$$

has poles at

$$\sqrt{\alpha^2 + \beta s} = \pm\,(2n - 1)\frac{\pi}{2}i$$

or

$$s_n = -\frac{1}{\beta}\left[(2n - 1)^2\,\frac{\pi^2}{4} + \alpha^2\right]$$

and its residues may be evaluated from the fact that residue at $s = s_n$ is given by

$$\frac{P(s_n)}{Q'(s_n)}e^{s_n t} = \frac{-2\alpha^2 \cosh\sqrt{\alpha^2 + \beta s_n}\,(1 - x/L)}{\beta\sqrt{\alpha^2 + \beta s_n}\,\sinh\sqrt{\alpha^2 + \beta s_n}}e^{s_n t}$$

$$= \frac{4\alpha^2}{\pi\beta}\,\frac{\cos\,(2n - 1)\frac{\pi}{2}(1 - x/L)}{(2n - 1)\,\sin\,(2n - 1)\frac{\pi}{2}}e^{s_n t}$$

$$= \frac{4\alpha^2}{\pi\beta} \frac{1}{2n - 1} \sin (2n - 1) \frac{\pi}{2} \frac{x}{L} e^{s_n t}$$

$$g(t) = \frac{4\alpha^2}{\pi\beta} \sum_{m = 1,3,--}^{\infty} \frac{1}{m} \sin m\frac{\pi}{2} \frac{x}{L} e^{-\theta_m t}$$

where

$$\theta_m = \frac{1}{\beta}\left(m^2\frac{\pi^2}{4} + \alpha^2\right)$$

From equation (6)

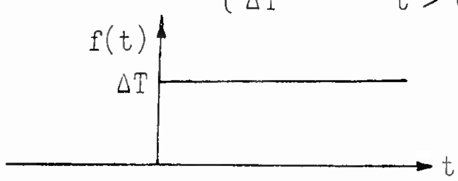
$$T(t,x) = \frac{4\alpha^2}{\pi\beta} \sum_{1,3,--}^{\infty} \frac{1}{m} \sin m\frac{\pi}{2} \frac{x}{L} e^{-\theta_m t} \int_0^t f(\lambda) e^{\theta_m \lambda} \, d\lambda$$

$$+ T_a + (T_b - T_a) \frac{\cosh \alpha(1 - x/L)}{\cosh \alpha} \qquad (7)$$

Specific Solutions

Response to Step Function

$$\text{Let } f(t) = \begin{cases} 0 & t < 0 \\ \Delta T & t > 0 \end{cases}$$



$$\int_0^t f(\lambda) e^{\theta_m \lambda} \, d\lambda = \Delta T \int_0^t e^{\theta_m \lambda} \, d\lambda = \frac{\Delta T}{\theta_m}(e^{\theta_m t} - 1)$$

$$T(t,x) = \Delta T \frac{4\alpha^2}{\pi} \sum_{1,3,--}^{\infty} \frac{1}{m\left(m^2\frac{\pi^2}{4} + \alpha^2\right)} \sin m\frac{\pi}{2}\frac{x}{L}\left(1 - e^{-\left(m^2\frac{\pi^2}{4} + \alpha^2\right)t/\beta}\right)$$

$$+ T_a + (T_b - T_a)\frac{\cosh \alpha(1 - x/L)}{\cosh \alpha} \tag{8}$$

The rate at which heat enters the base of the fin (per unit width) from the body being cooled is

$$\dot{Q}(t) = -\delta k \frac{\partial T}{\partial x}\bigg|_{x=0} = -\delta k \Delta T \frac{2\alpha^2}{L} \sum_{1,3,--}^{\infty} \frac{1}{m^2\frac{\pi^2}{4} + \alpha^2}\left[1 - e^{-\left(m^2\frac{\pi^2}{4} + \alpha^2\right)t/\beta}\right]$$

$$+ k\delta\frac{\alpha}{L}(T_b - T_a)\tanh \alpha$$

This can be written in a normalized form as

$$\dot{\mathcal{Q}}(t) = \frac{\dot{Q}(t) - \dot{Q}(0)}{\dot{Q}(\infty) - \dot{Q}(0)} = 1 - \frac{2\alpha}{\tanh \alpha} \sum_{1,3,--}^{\infty} \frac{1}{m^2\frac{\pi^2}{4} + \alpha^2} e^{-\left(m^2\frac{\pi^2}{4} + \alpha^2\right)t/\beta} \tag{9}$$

where use has been made of the relation

$$\sum_{1,3,--}^{\infty} \frac{1}{m^2\frac{\pi^2}{4} + \alpha^2} = \frac{1}{2}\frac{\tanh \alpha}{\alpha} \tag{10}$$

(This equation follows from equation 6.495-1 in reference (a); but it can also be obtained from the condition that if $\Delta T = T_b - T_a$, then $\dot{Q}(\infty) = 0$.)

$\dot{\mathcal{Q}}(t)$ is zero for $t = 0$ and approaches unity asymptotically as t ∞; also, except for values of t/β very close to zero, it can be expressed as a simple exponential rise

$$\dot{\mathcal{Q}}(t) \approx 1 - \frac{2\alpha}{\tanh \alpha}\frac{1}{\frac{\pi^2}{4} + \alpha^2} e^{-\left(\frac{\pi^2}{4} + \alpha^2\right)t/\beta}$$

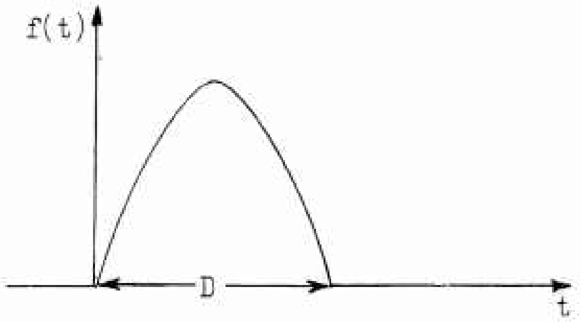
This equation is plotted in figure 1 for several values of α.

Thus the response of the cooling fin to a step change in ambient temperature can be simply characterized by a rise time, i.e., the time required for $\dot{\mathcal{Z}}(t)$ to reach some fraction, f, of its final value, unity. The rise time as calculated from the above equation is given in figure 2 for f = 80 and 90 percent.

Response to Sinusoidal Pulse

"Exact" Solution

$$\text{Let } f(t) = \begin{cases} \Delta T \sin \frac{\pi t}{D} & 0 \le t \le D \\ 0 & t \ge D \end{cases}$$



$$e^{-\theta_m t} \int_0^t \sin \frac{\pi\lambda}{D} e^{\theta_m \lambda} \, d\lambda = e^{-\theta_m t} e^{\theta_m \lambda} \frac{\left(\theta_m \sin \frac{\pi\lambda}{D} - \frac{\pi}{D} \cos \frac{\pi\lambda}{D}\right)}{\theta_m^2 + \pi^2/D^2} \Bigg|_0^t$$

$$= \frac{1}{\theta_m^2 + \frac{\pi^2}{D^2}} \left(\theta_m \sin \frac{\pi t}{D} - \frac{\pi}{D} \cos \frac{\pi t}{D} - \frac{\pi}{D} e^{-\theta_m t}\right).$$

$$e^{-\theta_m t} \int_0^D \sin \frac{\pi\lambda}{D} e^{\theta_m \lambda} \, d\lambda = \frac{\pi/D}{\theta_m^2 + \frac{\pi^2}{D^2}} \left[e^{-\theta_m (t - D)} + e^{-\theta_m t}\right]$$

$$T(t,x) = \frac{4}{\pi}\Delta T\alpha^2 r \sum_{1,3,--}^{\infty} \left\{ \frac{1}{m\left[\left(m^2\frac{\pi^2}{4} + \alpha^2\right)^2 + r^2\right]} \sin m\frac{\pi}{2}\frac{x}{L} \right.$$

$$\left. \times \left[\frac{m^2\frac{\pi^2}{4} + \alpha^2}{r} \sin\frac{\pi t}{D} - \cos\frac{\pi t}{D} + e^{-\frac{1}{r}\left(m^2\frac{\pi^2}{4} + \alpha^2\right)\frac{\pi t}{D}} \right] \right\}$$

$$+ T_a + (T_b - T_a)\frac{\cosh \alpha(1 - x/L)}{\cosh \alpha} \qquad t \leq D$$

$$= \frac{4}{\pi}\Delta T\alpha^2 r \sum_{1,3,--}^{\infty} \left\{ \frac{1}{m\left[\left(m^2\frac{\pi^2}{4} + \alpha^2\right)^2 + r^2\right]} \sin m\frac{\pi}{2}\frac{x}{L} \right.$$

$$\left. \times \left[e^{-\frac{1}{r}\left(m^2\frac{\pi^2}{4} + \alpha^2\right)\frac{\pi(t - D)}{D}} + e^{-\frac{1}{r}\left(m^2\frac{\pi^2}{4} + \alpha^2\right)\frac{\pi t}{D}} \right] \right\}$$

$$+ T_a + (T_b - T_a)\frac{\cosh \alpha(1 - x/L)}{\cosh \alpha} \qquad t \geq D \qquad (11)$$

where

$$r = \frac{\pi\beta}{D}$$

From $\dot{Q}(t) = - k\delta\frac{\partial T}{\partial x}\Big|_{x = 0}$

$$\frac{L}{k\delta}\dot{Q}(t) = -2\Delta T\alpha^2 r \sum_{1,3,--}^{\infty} \left\{ \frac{1}{\left(m^2\frac{\pi^2}{4} + \alpha^2\right)^2 + r^2} \right.$$

$$\left. \times \left[\frac{m^2\frac{\pi^2}{4} + \alpha^2}{r} \sin\frac{\pi t}{D} - \cos\frac{\pi t}{D} + e^{-\frac{1}{r}\left(m^2\frac{\pi^2}{4} + \alpha^2\right)\frac{\pi t}{D}} \right] \right\}$$

$$+ \alpha(T_b - T_a) \tanh\alpha \qquad\qquad t \leq D$$

$$= -2\Delta T\alpha^2 r \sum_{1,3,--}^{\infty} \left\{ \frac{1}{\left(m^2\frac{\pi^2}{4} + \alpha^2\right)^2 + r^2} \right.$$

$$\left. \times \left[e^{-\frac{1}{r}\left(m^2\frac{\pi^2}{4} + \alpha^2\right)\left(\frac{\pi t}{D} - \pi\right)} + e^{-\frac{1}{r}\left(m^2\frac{\pi^2}{4} + \alpha^2\right)\frac{\pi t}{D}} \right] \right\}$$

$$+ \alpha(T_b - T_a) \tanh\alpha \qquad\qquad t \geq D \qquad\qquad (12)$$

<u>Quasi-Static Approximation</u>

From the temperature distribution for constant ambient temperature, as given by equation (4), the rate of heat flow into base of the fin under this condition is

$$\dot{Q} = -k\delta\frac{dT}{dx}\bigg|_{x=0} = k\delta\frac{\alpha}{L}(T_b - T_a)\tanh\alpha$$

In accordance with the discussion in the Introduction an approximate solution for variable ambient temperature can be written by replacing T_a by $T_a(t)$. In the case where

$$T_a(t) = \begin{cases} T_a + \Delta T \sin\frac{\pi t}{D} & \text{for } t \leq D \\ 0 & \text{for } t \geq D \end{cases}$$

the approximate solution becomes

$$\frac{L}{k\delta}\dot{Q}(t) = -\alpha \tanh\alpha\ \Delta T \sin\frac{\pi t}{D} + (T_b - T_a)\,\alpha \tanh\alpha \qquad t \le D$$

$$= (T_b - T_a)\,\alpha \tanh\alpha \qquad t \ge D \qquad (13)$$

DISCUSSION

The change in cooling rate produced by a change in ambient temperature in the form of a sinusoidal pulse was calculated from equations (12) and (13) for three fins of the dimensions given below.

	Fin 1	2	3
L (ft)	0.0625	0.110	0.194
δ (ft)	0.0015	0.0021	0.00327

The assumed physical constants are:

$h = 1$ btu/(hr ft^2 °F)

$k = 100$ btu/(hr ft °F)

$\rho = 168$ lb/ft^3

$C_p = 0.23$ btu/lb °F

and the temperatures involved are

$T_b = 150°$ F

$T_a = 95°$ F

$\Delta T = 50°$ F

The same three values of pulse duration were chosen for each fin.

D (hr) = 0.22, 0.0338, and 0.00375.

The derived constants are:

	Fin 1	2	3
α	0.228	0.340	0.480
β (hr)	0.00151	0.00468	0.01458
(90%) t_r (hr)	0.00125	0.00378	0.0112

The results of these calculations are shown in figures 3, 4, and 5. The ratio of the instantaneous rate of heat flow into the base of the fin to the steady state rate is plotted versus the ratio of time to the endurance of the pulse.

The ratio of the pulse duration to the rise time of the fin (D/t_r) is indicated on each curve. This ratio is a measure of the "slowness" of the ambient temperature change, and, as seen from the figures and as is intuitively obvious, the exact solution approaches the quasi-static approximation for large values of D/t_r.

The ambient temperature time-history given in figure 6 of reference (b) may be approximated by a sinusoidal pulse of duration D = 0.22 hour. Thus curve A of figure 3 gives the exact solution for a problem analogous to the one considered in reference (b). In this case use of the approximate solution is very satisfactory, and the small errors it does produce are nearly all on the conservative side.

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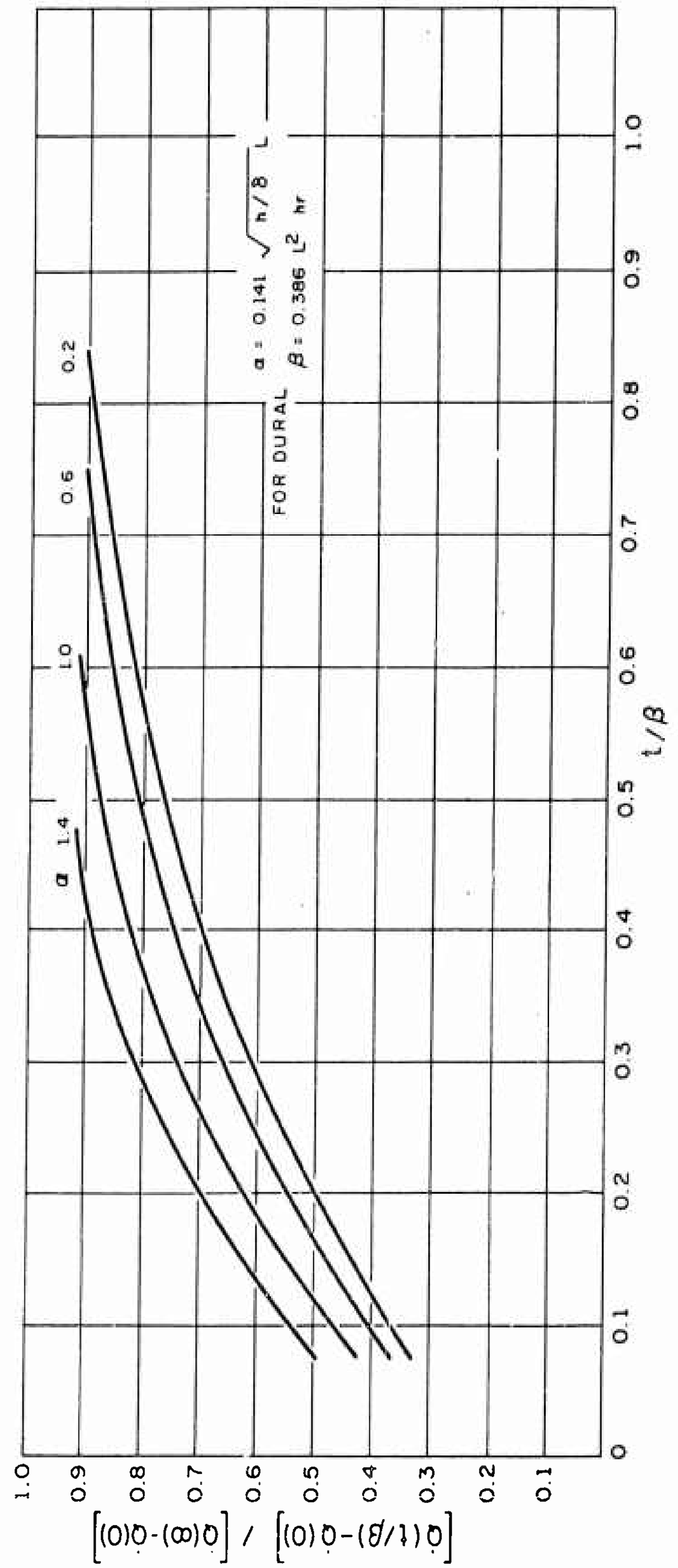

FIGURE 1 - Response to Step Function

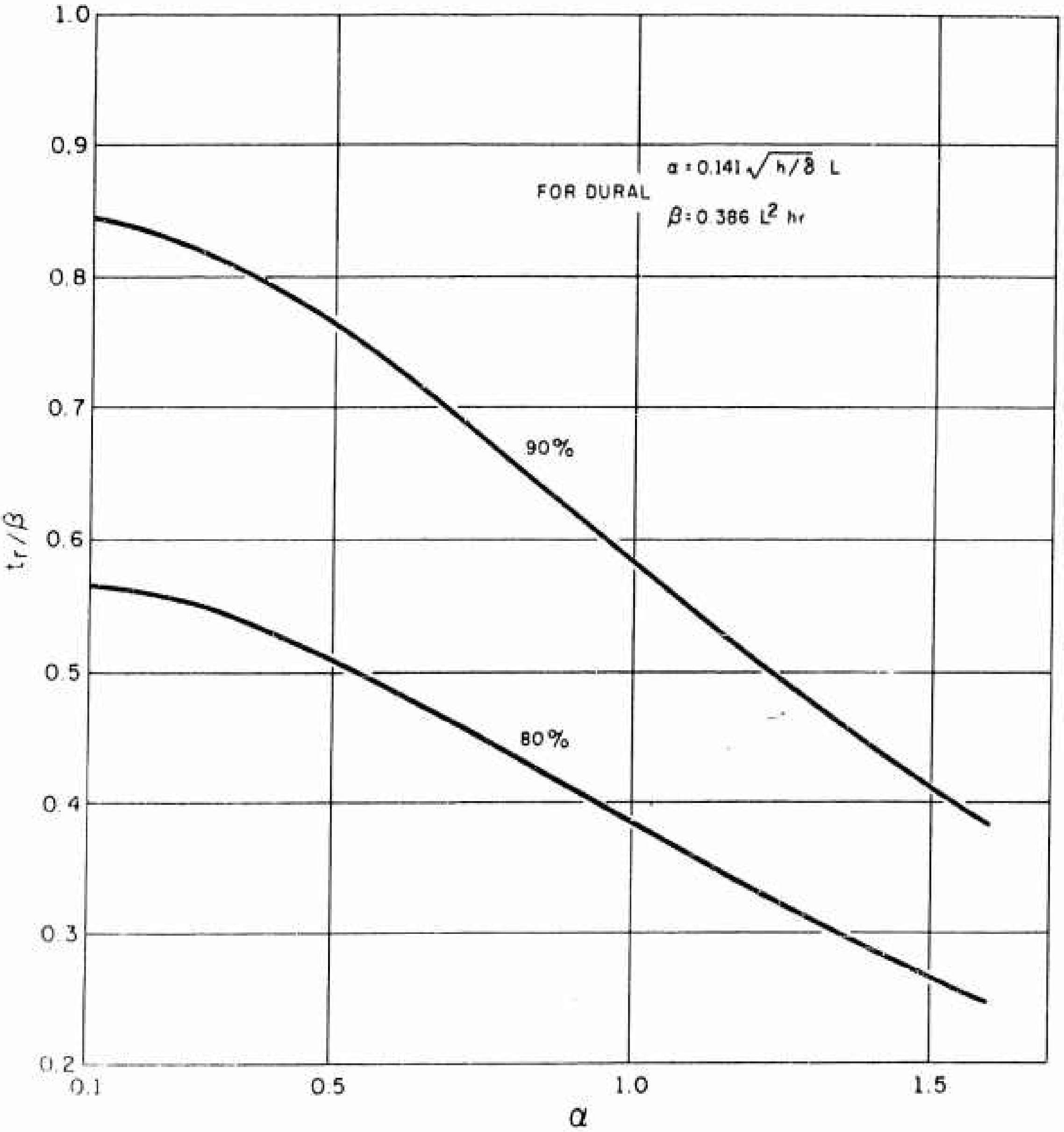

FIGURE 2 - Rise Time

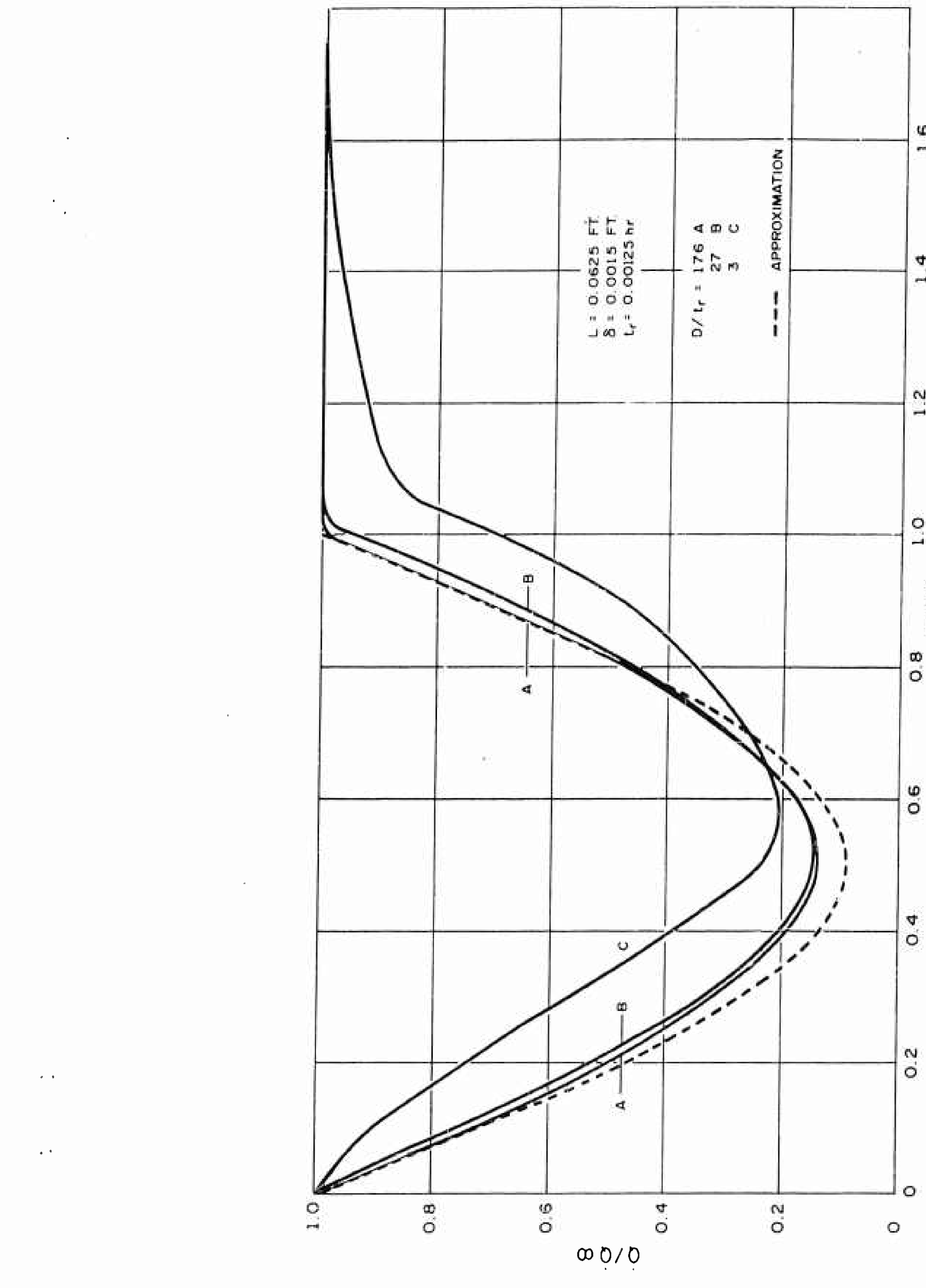

FIGURE 3 - Fin 1 Heat Flow Rate

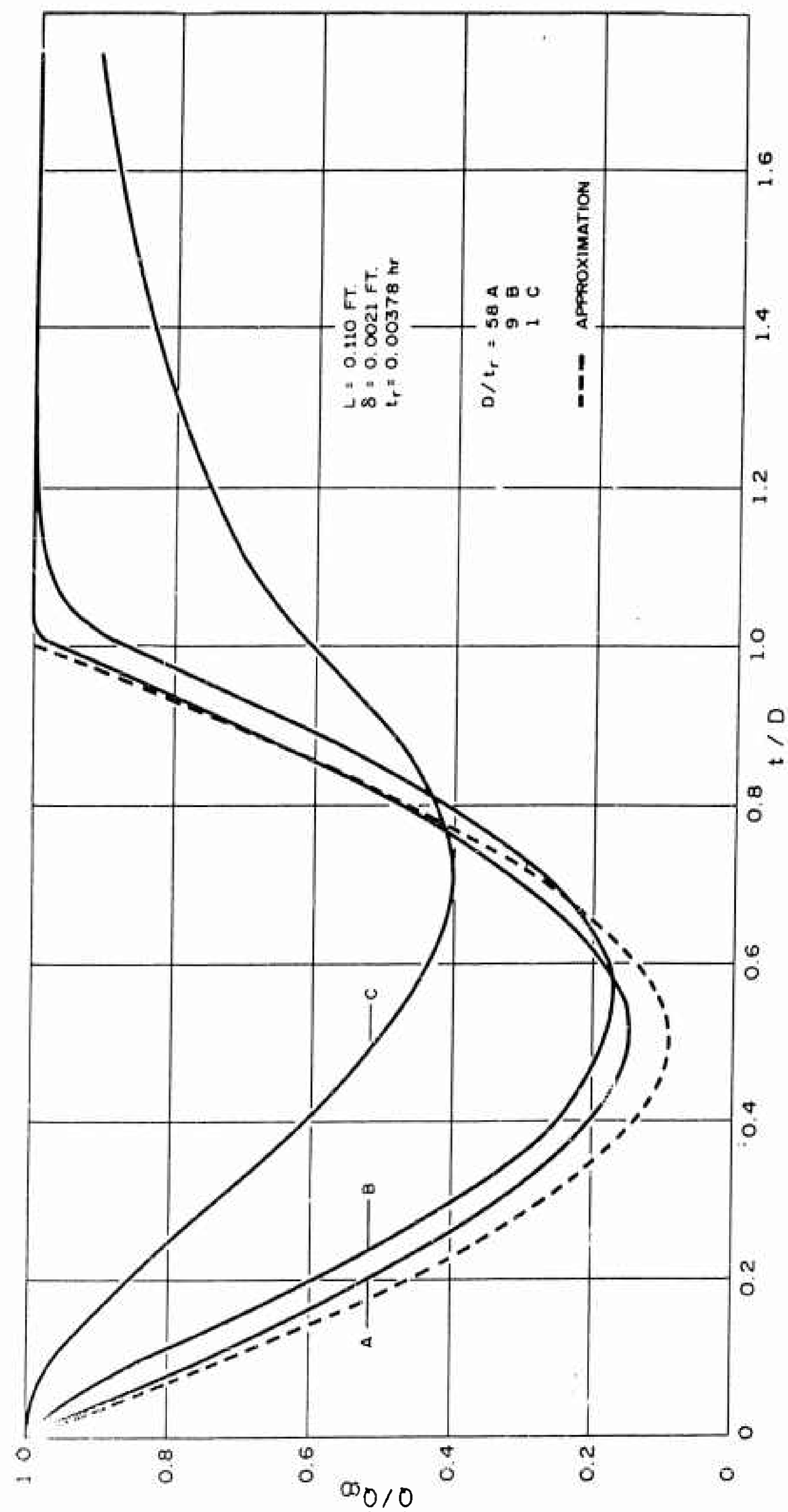

FIGURE 4 - Fin 2 Heat Flow Rate

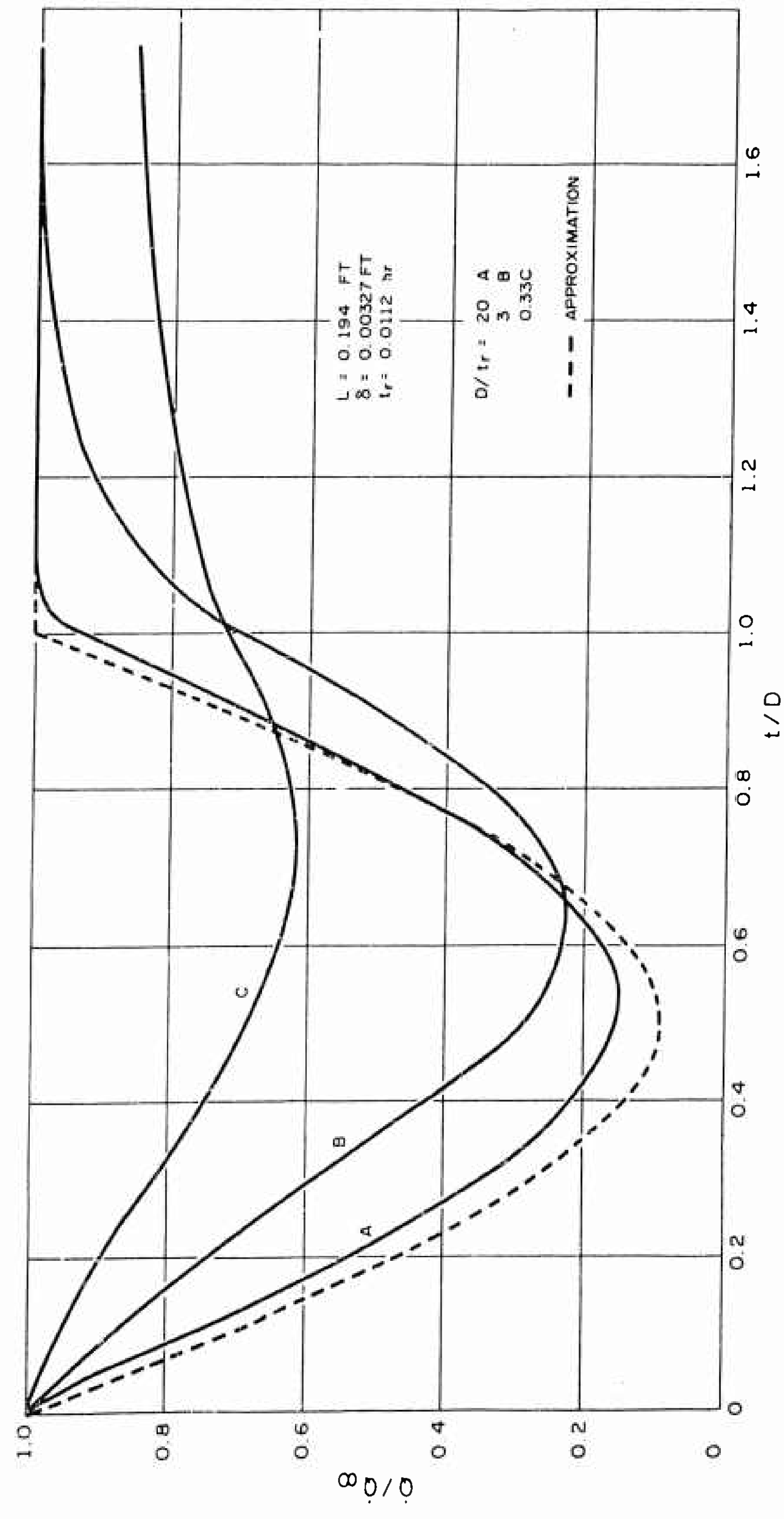

FIGURE 5 - Fin 3 Heat Flow Rate

APPENDIX A

LIST OF SYMBOLS

L	length of fin
δ	thickness (smallest dimension, sometimes called "width" in heat transfer literature)
ρ	density
C_p	specific heat
k	thermal conductivity
h	heat transfer coefficient
t	time
t_r	rise time
T	temperature
ΔT	temperature increment
T_b	base temperature
T_a	constant ambient temperature
$T_a(t) = T_a + f(t)$	varying ambient temperature
$f(t)$	variable component of ambient temperature
D	duration of sinusoidal pulse
$\alpha \equiv \sqrt{2hL^2/\delta k}$	dimensionless constant
$\beta \equiv C_E \rho L^2/k$	constant (dimension of time)
$r \equiv \pi\beta/D$	dimensionless constant

APPENDIX B

SUMMATION OF SERIES

In equations (11) and (12) the coefficient of $\sin \frac{\pi t}{D}$ is proportional to

$$\frac{1}{r} \sum_{1,3,--}^{\infty} \frac{m^2 \frac{\pi^2}{4} + \alpha^2}{(m^2 \frac{\pi^2}{4} + \alpha^2)^2 + r^2}$$

For large values of r the calculation of many terms in this series is required for accuracy; however, this series can be summed exactly. It is convenient to evaluate simultaneously the series

$$\sum_{1,3,--}^{\infty} \frac{1}{(m^2 \frac{\pi^2}{4} + \alpha^2)^2 + r^2}$$

appearing in the coefficient of $\cos \frac{\pi t}{D}$.

Let

$$A = \frac{1}{r} \sum_{1,3,--}^{\infty} \frac{m^2 \frac{\pi^2}{4} + \alpha^2}{(m^2 \frac{\pi^2}{4} + \alpha^2)^2 + r^2}$$

$$B = \sum_{1,3,--}^{\infty} \frac{1}{(m^2 \frac{\pi^2}{4} + \alpha^2)^2 + r^2}$$

$$A + iB = \frac{1}{r} \sum \frac{m^2 \frac{\pi^2}{4} + \alpha^2 + ir}{(m^2 \frac{\pi^2}{4} + \alpha^2) + r^2}$$

$$= \frac{1}{r} \sum \frac{1}{m^2 \frac{\pi^2}{4} + \alpha^2 - ir}$$

But

$$\sum_{1,3,--}^{\infty} \frac{1}{m^2\frac{\pi^2}{4} + w^2} = \frac{1}{2}\frac{\tanh w}{w} \qquad \text{(See equation (10))}$$

$$\therefore A + iB = \frac{1}{2r}\frac{\tanh\sqrt{\alpha^2 - ir}}{\sqrt{\alpha^2 - ir}}$$

Write

$$\sqrt{\alpha^2 - ir} = Z = x + iy \qquad (x,\ y \text{ real})$$

$$\frac{\tanh Z}{Z} = \frac{1}{Z}\frac{e^Z - e^{-Z}}{e^Z + e^{-Z}}$$

$$= \frac{\overline{Z}}{|Z|^2}\frac{e^Z - e^{-Z}}{e^Z + e^{-Z}}\frac{e^{\overline{Z}} + e^{-\overline{Z}}}{e^{\overline{Z}} + e^{-\overline{Z}}}$$

$$= \frac{x - iy}{x^2 + y^2}\frac{e^{2x} - e^{-2x} + e^{2iy} - e^{-2iy}}{e^{2x} + e^{-2x} + e^{2iy} + e^{-2iy}}$$

$$= \frac{x - iy}{x^2 + y^2}\frac{\sinh 2x + i \sin 2y}{\cosh 2x + \cos 2y}$$

$$A + iB = \frac{1}{2r}\frac{x \sinh 2x + y \sin 2y + i(x \sin 2y - y \sinh 2x)}{(x^2 + y^2)(\cosh 2x + \cos 2y)}$$

From

$$\sqrt{\alpha^2 - ir} = x + iy$$

$$x^2 - y^2 = \alpha^2$$

$$2xy = - r$$

$$\therefore x = \pm\sqrt{\frac{\alpha^2 + \sqrt{\alpha^4 + r^2}}{2}}, \quad y = \mp\sqrt{\frac{-\alpha^2 + \sqrt{\alpha^4 + r^2}}{2}}$$

$$x^2 + y^2 = \sqrt{\alpha^4 + r^2}$$

$$\therefore A = \frac{1}{r}\sum_{1,3,--}^{\infty} \frac{m^2\frac{\pi^2}{4} + \alpha^2}{\left(m^2\frac{\pi^2}{4} + \alpha^2\right)^2 + r^2} = \frac{1}{2r\sqrt{\alpha^4 + r^2}}\;\frac{u \sinh 2u + v \sin 2v}{\cosh 2u + \cos 2v}$$

$$B = \sum_{1,3,--}^{\infty} \frac{1}{\left(m^2\frac{\pi^2}{4} + \alpha^2\right)^2 + r^2} = \frac{1}{2r\sqrt{\alpha^4 + r^2}}\;\frac{v \sinh 2u - u \sin 2v}{\cosh 2u + \cos 2v}$$

where

$$u = \sqrt{\frac{\alpha^2 + \sqrt{\alpha^4 + r^2}}{2}}, \quad v = \sqrt{\frac{-\alpha^2 + \sqrt{\alpha^4 + r^2}}{2}}$$